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## ON THE THEORY OF FUNCTIONS OF A TRIPLE VARIABLE.*

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### §1. THE ALGEBRA DEFINED.

Since single real numbers may be represented by the points of a linear continuum we may speak of their algebra as that of one-dimensional extent. Similarly the algebra of double numbers (or number pairs or complex numbers) finds its geometrical image in the points of a surface, or a two-dimensional extent. But no attempt has ever succeeded in defining a system of triple numbers whose algebra is entirely analogous to that of one-dimensional and two-dimensional extent. In fact it has been shown† that no triple algebra exists at the same time conserving all the formal laws of the fundamental operations as developed in our ordinary algebra and also maintaining the important theorem: *If a product of two factors equals zero one at least of the factors is zero.* This theorem plays a fundamental rôle in algebraic analysis and no complete surrender of it can be permitted in constructing a new algebra.

Nevertheless, in attempting to devise a system of units such that numbers built up from them may be imaged in three-dimensional extent, it is just this law which I have agreed to surrender. The problem is then to find a system of units such that the additional complication incident to the surrender of this law shall be reduced to a minimum. Several systems of units with their laws were assumed arbitrarily, but their algebra in every case presented almost insurmountable difficulties. Finally by an analysis of commutative operations in space—translations and rotations—the system of units here employed was defined, first geometrically and then algebraically. We are now concerned with the latter alone.

We begin by defining a system of units by the following equations:

$$\text{I.}\quad r.r=r,\ \frac{r}{r}=r,\ ri=ir=i,\ i^2+r=0,\ s.s=s,\ \frac{s}{s}=s,\ r.s=s.r=i.s=s.i=0.$$

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*Read before the American Mathematical Society, October 31, 1908.

†See Stolz and Gmeiner's *Theoretische Arithmetik.*

A number in this system is represented in the form

$$ar+bi+cs,$$

where $a$, $b$, $c$ are ordinary reals. It may be looked on as defining a point in space of which the Cartesian coordinates (in a rectangular system) are $a$, $b$, $c$. A more fruitful view of its geometric significance, however, is the following: A plane of reference (chosen at will, but for convenience spoken of as the horizontal plane) has in it rectangular axes intersecting at $O$. Then the point denoted by $ar+bi+cs$ is $c$ units in perpendicular distance from the plane and has as its projection on the plane the point which in ordinary complex quantities is denoted by $a+bi$. Then every number in this system corresponds uniquely to a point in space; and, conversely, to every point in space there corresponds uniquely a number of this system. The projection on the plane of reference of the general point $ar+bi+cs$ will be denoted not by $ar+bi$ but by

$$ar+bi+0s.$$

We reserve $ar+bi$ for another meaning. It may conveniently be taken to represent a line through the point $ar+bi+0s$ and perpendicular to the plane of reference. It thus includes all points $ar+bi+zs$ where $z$ is a real variable.

Operational and other definitions will follow closely the analogy of the ordinary complex numbers. We define thus the necessary and sufficient condition of equality:

If $\qquad ar+bi+cs=\alpha r+\beta i+\gamma s,$
then $\qquad a=\alpha,\ b=\beta,\ c=\gamma.$

Addition and subtraction are defined by the equations:

II. Addition: $(ar+bi+cs)+(\alpha r+\beta i+\gamma s)=(a+\alpha)r+(b+\beta)i+(c+\gamma)s.$
III. Subtraction: $(ar+bi+cs)-(\alpha r+\beta i+\gamma s)=(a-\alpha)r+(b-\beta)i+(c-\gamma)s.$

The geometric interpretation is clear. Moreover, these operations obey the usual laws.

If we actually perform the operation of multiplication of two triple numbers, observing the laws of combining the units defined in I, we obtain our multiplication formula, thus:

IV. Multiplication: $(ar+bi+cs)(\alpha r+\beta i+\gamma s)$
$$=(a\alpha-b\beta)r+(a\beta+b\alpha)i+c\gamma s.$$

Adopting the usual definition of division as the inverse of multiplication, we have readily:

V. Division: $\dfrac{ar+bi+cs}{\alpha r+\beta i+\gamma s}=\dfrac{a\alpha+b\beta}{\alpha^2+\beta^2}r+\dfrac{-a\beta+\alpha b}{\alpha^2+\beta^2}i+\dfrac{c}{\gamma}s.$

That the formal laws of ordinary algebra hold for multiplication and division is evident from the form of the results in IV and V.

So far we have not brought our new numbers into relation with ordinary real numbers. The definitions of addition and multiplication enable us to do this. By the formula for addition we have

$$m(ar+bi+cs)=mar+mbi+mcs.$$

The multiplication formula gives the same result for the product,

$$(ar+bi+cs)(mr+0i+ms)=mar+mbi+mcs.$$

Hence we shall define

VI. $$m=mr+0i+ms,$$

and all real numbers are to be combined with triple numbers in accordance with this definition.

We have now to develop the zero-product theorem. Set

$$(ar+bi+cs)(\alpha r+\beta i+\gamma s)=(a\alpha-b\beta)r+(a\beta+b\alpha)i+c\gamma s=0.$$

This requires that

$$a\alpha-b\beta=0,\quad a\beta+b\alpha=0,\quad c\gamma=0.$$

There is no difficulty in verifying that these relations can be satisfied only when one of the factors is of the form $0r+0i+0s$ or when one of them is of the form $0r+0i+cs$ and the other is of the form $\alpha r+\beta i+0s$. We have then the following theorem:

THEOREM. *If the product of two factors is zero, then one of them is zero; or, one of them is of the form* $0r+0i+cs$ *while the other is of the form* $\alpha r+\beta i+0s$.

There are thus in this system three quantities playing in some respects the rôle of a zero.

It is now easy to see that division is unique and determinate *except in three cases,** namely,

VII. $$\frac{0r+0i+0s}{0r+0i+0s}, \quad \frac{0r+0i+cs}{0r+0i+\gamma s}, \quad \frac{ar+bi+0s}{\alpha r+\beta i+0s}.$$

The first is the indeterminate 0/0 of our ordinary complex algebra. The other two have an easy geometric interpretation. For the second, the distance of the point from the plane is fixed, but its projection on the plane is indeterminate. For the third, the projection on the plane of reference is fixed, but the distance from the plane is indeterminate.

If one wishes to interpret for ordinary complex algebra any results obtained by the use of this triple algebra he has merely to omit the terms containing $s$ and to set $r=1$. The formulae and results then become those of ordinary complex algebra.

Our new algebra is now completely defined and related to our former algebra. It is now apparent that *the only added complication or difficulty which can possibly arise will be due to the existence of three indeterminates in division instead of one as in the former algebra.* That this is by no means an insurmountable difficulty becomes apparent as our work progresses.

## §2. FUNCTIONAL DEPENDENCE.

We shall employ the notation

$$R \equiv xr+yi+zs,$$
$$S \equiv ur+vi+ws,$$

and shall say that $S$ is a function of $R$ when $S$ may be obtained by performing any set of operations upon $R$ considered as a single whole. This of course is precisely analogous to the ordinary conception of a function of a complex variable. Moreover we shall further assume that the functions with which we deal have a diffierential coefficient.

Then if $S=f(R)$, where the form of $f$ is not specified, we have

$$ur+vi+ws=S=f(R)=f(xr+yi+zs).$$

Differentiating with respect to $x$, $y$, $z$, we obtain

$$(1) \qquad \frac{\partial S}{\partial x}=\frac{dS}{dR}\frac{\partial R}{\partial x}=rf'(R)=r\frac{dS}{dR},$$

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*Of course this presupposes the introduction of infinity. This is not expressly defined here because it introduces nothing new.

$$(2) \qquad \frac{\partial S}{\partial y}=\frac{dS}{dR}\frac{\partial R}{\partial y}=if'(R)=i\frac{dS}{dR},$$

$$(3) \qquad \frac{\partial S}{\partial z}=\frac{dS}{dR}\frac{\partial R}{\partial z}=sf'(R)=s\frac{dS}{dR}.$$

These equations do not determine any relations existing between *w* on the one hand and either *u* or *v* or both of them on the other. This was to be expected from the definitions of the fundamental operations. In view of these definitions it is evident that

$$w=f(z).$$

But there is a necessary relation between *u* and *v*, and it may be developed from the first two differential equations above. We have

$$\frac{\partial S}{\partial x}=r\frac{\partial S}{\partial R}, \quad \frac{\partial S}{\partial y}=i\frac{\partial S}{\partial R}.$$

Multiplying the first by *r* and the second by *i* and remembering that $i^2=-r$, we obtain

$$r\frac{\partial S}{\partial x}=-i\frac{\partial S}{\partial y}.$$

Now the relation between *u* and *v* is independent of *w*, and hence in order to find that relation we may ignore *w* altogether. Geometrically, it amounts to this: For the present we are concerned with the relations between the coordinates of the projections of the variable point on the plane of reference. These are entirely independent of the point's distance from the plane; that is, of *w*. Therefore from the last equation we may write:

$$r\frac{\partial}{\partial x}(ur+vi)=-i\frac{\partial}{\partial y}(ur+vi).$$

$$\therefore \frac{\partial u}{\partial x}r+\frac{\partial v}{\partial x}i=-\frac{\partial u}{\partial y}i+\frac{\partial v}{\partial y}e. \qquad \therefore \frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}.$$

These are necessary relations between *w* and *v* regarded as functions of *x* and *y*. By a method entirely analogous to that employed in the ordinary theory of complex variables they may be shown to be sufficient conditions. Then we have the theorem:

THEOREM I. *In order that we may have*

$$ur+vi+ws=S=f(R)=f(xr+yi+zs)$$

*it is necessary and sufficient that the following relations exist:*

$$w=f(z),\quad \frac{\partial u}{\partial x}=\frac{\partial v}{\partial y},\quad \frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}.$$

Adding equations (1) and (3) we have

$$\frac{dS}{dR}r+\frac{dS}{dR}s=\frac{\partial S}{\partial x}+\frac{\partial u}{\partial z}.$$

Employing equation VI of the preceding section, this becomes

$$\frac{dS}{dR}=\frac{\partial S}{\partial x}+\frac{\partial S}{\partial z}. \tag{4}$$

(It is evident that VI holds not only when $m$ is real but also when $m$ is a triple number.) The second member of (4) is evidently independent of the direction in which the point $R+\triangle R$ approaches $R$; hence

THEOREM II. *The value of the ratio* $\frac{dS}{dR}$ *is independent of the direction in space in which* $R+\triangle R$ *approaches* $R$.

When in

$$ur+vi+ws=S=f(R)=f(xr+yi+zs)$$

the form of $f$ is given, it clearly suffices theoretically to determine $u$, $v$, $w$. We shall show conversely that if any one of the three quantities $u$, $v$, $w$ is given, this alone is sufficient to determine $f$ and the other two of the functions $u$, $v$, $w$.

First, suppose $w$ is given. It must be a function of $z$ alone such that $w=f(z)$, and hence the form of $f$ is determined. It follows as a consequence that $u$ and $v$ have perfectly definite forms.

On the other hand we have only to notice that the relation expressed between $u$ and $v$ in theorem I is exactly that which in the theory of complex variables suffices to determine one of them when the other is given. With both of them determinate, it is evident that the form of $f$ is fixed and therefore $w$ is perfectly definite. Hence,

THEOREM III. *If any one of the functions* $u$, $v$, $w$ *(as* $u$*) is given in terms of* $x$, $y$, $z$, *or any of them, this alone suffices to determine the other two functions (as* $v$ *and* $w$*) and the form of* $f$.